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Feynman graph derivation of the Einstein quadrupole formula

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Abstract. The one-graviton transition operator and, consequently, the classical energy-loss formula for gravitational radiation are derived from the Feynman graphs of helicity ±2 theories of gravitation. The calculations are done both for the case of electromagnetic and gravitational scattering. The departure of the in and out states from plane waves owing to the long-range nature of gravitation is taken into account to improve the Born approximation calculations. This also includes a long-range modification of the graviton wavefunction which is shown to be equivalent to the classical problem of the true light cones deviating logarithmically at large distances from the flat space light cones. The transition from the *S*-matrix elements calculated graphically to the graviton transition operator is done by using complementarity of space–time and momentum descriptions. The energy-loss formula derived originally by Einstein is shown to be correct.

1. Introduction

The problem of gravitational radiation has had a controversial history right from the early days of general relativity. The magnitude of the controversy may be visualised by the fact that even its existence was intensely debated by Einstein and Robertson for a long time. In his 1918 article Einstein derived the following formula for the power emitted by gravitational radiation

$$\frac{\mathrm{d}E_g}{\mathrm{d}t} = -\frac{G}{8\pi}|\varepsilon_{ij}(\lambda, \hat{n})\dddot{D}^{ij}|^2\,\mathrm{d}\Omega_{\hat{n}} \tag{1.1}$$

where G is the Newtonian constant, $\varepsilon_{ij}(\lambda, \hat{n})$ is the polarisation tensor for the mode λ propagating along $\hat{n}$ in the gauge $\varepsilon^{0i} = \varepsilon^{00} = 0$ and D^{ij} is the quadrupole moment for the material system.

Nearly half a century later this result has been criticised by a number of authors (Ehlers *et al* 1976) on the grounds of self-consistency. The derivation mentioned above was essentially based on reducing Einstein's equations to the quasi-linear form

$$\Box_g g = T(m, g) + \Lambda(g) \tag{1.2}$$

after fixing the coordinate choice according to some conditions

$$C(g) = 0. \tag{1.3}$$

In (1.2) $\Box_g$ is the Laplacian defined on space–time with metric g, $T(m, g)$ is a functional of the metric as well as the matter variables m, and $\Lambda(g)$ is a functional of g only. In

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general it is very difficult, if not impossible, to solve (1.2). The strategy one adopts usually consists in solving (1.2) iteratively. This is done by computing a sequence ${}^1g, {}^2g, \ldots, {}^Ng$ (where each member is a functional of the matter fields m) and then imposing the conditions (1.3). Equation (1.3) for ${}^Ng$ is then equivalent to the $(N-1)$th-order equations of motion for the matter variables, i.e.

$$ {}^{N-1}\nabla \cdot T(m, {}^{N-1}g) = 0 \sim C({}^Ng) = 0. \tag{1.4}$$

Since the field equations and equations of motion for matter are not independent in general relativity, the above procedure is necessary for self-consistency.

Einstein derived (1.1) by linearising the field equations. Up to this approximation, the equation of motion for matter must be

$$ {}^0\nabla \cdot T(m, {}^0g) = 0. \tag{1.5}$$

For a system of point particles not acted upon by any non-gravitational forces like electromagnetism, etc, (1.5) implied $\mathrm{d}u^\mu/\mathrm{d}\tau = 0$ and consequently $\dddot{D}^{ij} = 0$. Thus the radiation intensity in the linearised approximation vanishes when the particles are not acted upon by any non-gravitational forces! In electrodynamics such a difficulty does not arise because the current is conserved for all equations of motion of the charged particle.

If the particles are acted upon by non-gravitational forces ${}^0T(m, {}^0g)$ is not merely the stress tensor of the particles and (1.5) does not imply $\mathrm{d}u^\mu/\mathrm{d}\tau = 0$ (for example $\mathrm{d}u^\mu/\mathrm{d}\tau = eF^{\mu\nu}\mu_\nu$ for electromagnetism) and hence $\dddot{D}_{ij} \neq 0$ and (1.1) is consistently derivable in the linear approximation.

Despite the impression created by Einstein (1918) that (1.1) was derived in the linear approximation it is not clear that it is really so as in evaluating the radiation potentials Einstein *implicitly* uses not (1.5) but

$$ {}^1\nabla \cdot T(m, {}^1g) = 0 \tag{1.6}$$

which indeed ensures non-vanishing $\dddot{D}_{ij}$. But according to (1.4) g must be computed to at least ${}^2g$ which is clearly beyond the linear approximation. The question is then raised as to whether all effects have been consistently included to the desired approximation.

Ehlers *et al* (1976) have urged a closer inspection of (1.1) in view of the above difficulties. They have also listed a number of pitfalls to be carefully avoided for a satisfactory derivation of (1.1). Further, the 1918 derivation and many subsequent derivations based on it (Weinberg 1972) use the 'pseudo-tensor' to relate the outgoing flux to the radiation potentials. It is therefore desirable to find methods that do not make explicit use of the pseudo-tensor.

Recently Rosenblum (1978) has calculated the energy loss during gravitational scattering of two point particles of equal mass in the fast approximation scheme. He concludes that (1.1) is wrong by a factor of nearly 2.5! Rosenblum deals directly with the equations of motion as opposed to calculating the energy loss at infinity.

Cooperstock and Hobill (1979) have initiated a series of investigations into the radiation problem wherein they make a model for the radiating system which consists of two masses held initially by a strut which is later disrupted. They conclude that the quadrupole formula is incorrect and that there are large structure-dependent corrections to it. In the case of the binary pulsar PSR 1913 + 16 (Taylor *et al* 1979) such corrections amount to factors of several orders of magnitude and these are completely ruled out by present data.

Thus the status of what we term 'classical derivations' of the quadrupole formula is very confusing and it is perhaps safe to say that many of the approximations found in the literature are under shadows of doubt.

Motivated by these considerations we decided to take a fresh look at the problem of gravitational radiation. In order to avoid the various pitfalls cited by Ehlers *et al* (1976) we decided to approach the problem from the quantum gravity point of view. Normally one would not resort to quantum methods to calculate the energy loss from macroscopic systems, even though there is nothing wrong in principle in doing so. But in the case of gravitational radiation the classical calculations, as argued before, have been put under a shadow. We are not implying, of course, that purely classical remedies to these difficulties do not exist. (After our work was completed, a number of papers have appeared which have sought classical remedies to the issues raised in our paper, e.g. Papapetrou and Linet (1980), Breuer and Eckart (1981), Andersson (1980) and Walker and Will (1980).)

It is our feeling, and we hope to justify this in our article, that many of these difficulties can be handled in a more lucid and tractable fashion in the quantum approach even though a one-to-one correspondence with the difficulties of the classical calculations is hard to establish. Of course, one may right at the onset question the wisdom of using quantum gravity methods when a fully consistent quantum theory of gravitation is yet to be developed. Our answer to this is that we are ultimately interested only in the classical limit of our calculations and it is well known that classical results are fully reproduced in any field theory by the so-called tree diagrams and for these, consistent rules of calculation, namely Feynman graph perturbation expansions, do exist for quantum gravity.

We pause at this stage to enumerate a few of the obvious advantages of the diagrammatic approach of quantum gravity over its classical counterpart. First of all, the particle equations of motion are automatically taken care of in terms of energy–momentum conservation laws at each vertex; the analogue of general coordinate invariance of general relativity translates to gauge invariance in quantum gravity and the latter is computationally easier to realise than the former; in the tree graph approximation no renormalisation is necessary and thus one can even treat point particles without any need for renormalisation in contrast to the classical calculations where point particles represent singular sources and their treatment requires careful renormalisation procedures; no need for the introduction of the pseudo-tensor, etc.

However, just as in the case of electrodynamics the long-range nature of gravitational interactions introduce certain difficulties in a consistent evaluation of the S-matrix elements. These difficulties are overcome by replacing the usual plane-wave in and out states by the appropriate Coulomb-like modified states. We have assumed that the dominant non-perturbative modifications at large distances are caused by the $1/r$ potential. Even in the classical general relativistic treatments this difficulty persists in the following sense: even at very large distances from gravitating objects the paths of light rays deviate from their flat paths logarithmically. The justification, in the quantum approach, for modifying the in (out) states only due to the $1/r$ potential is again to be found in the equations of propagation of gravitational perturbances in a Schwarzchild background of general relativity (C V Visveshwara, private communication), where the non-perturbative aspect (occurrence of $\ln r$ terms) is seen to be entirely due to the $1/r$ potential. At this stage a criticism often voiced against Feynman graph calculations on the grounds that the parameter Gm_1m_2 could in general be very large and perturbation theory should break down badly (Thorné *et al* 1975) should be pointed out. It is shown in this paper that no such thing happens, at least for the classical limit. The reason for this is that no S-matrix element in the tree graph approximation depends on the parameter Gm_1m_2 itself; rather, the dependence is on parameters like Gmq where q is the momentum transfer and this translates to the familiar small parameter of general relativity, i.e. GM/Rc^2 in space–time language. It is a remarkable feature that even though the modified in (out) states explicitly involve the large parameter Gm_1m_2 (hence the need for non-perturbative treatment of these states) none of the observables (S-matrix elements) have this explicit dependence on Gm_1m_2.

We emphasise here that Born approximation amplitudes for graviton emission have been calculated by a number of authors before (Barker *et al* 1969, Barker and Gupta 1974, Gupta and Redford 1979). But they computed quantities like the differential cross section (which is now a function of several variables as opposed to the scattering situation where it depends on only one momentum transfer variable), or the frequency spectrum, or the total energy loss (at all impact parameters), etc. The energy loss has also been computed using frequency cut-off, etc. In some cases the above calculations have also been performed in approximations that go beyond the Born approximation. But in all these calculations the precise form of the classical quadrupole formula inclusive of its numerical features is very hard to recognise.

As argued before, the correct S-matrix elements are those that have been calculated on the basis of the distorted in (out) states and these amplitudes are extremely hard to evaluate explicitly. It is at this stage that we introduce an innovative method to extract the classical limit that is the main feature of this paper. Our approach is to identify directly the one-graviton transition operator from the structure of the S-matrix elements. This we do for two separate physical situations, namely two particles of unequal mass scattering (i) gravitationally and (ii) electromagnetically. The relevant graviton transition operator is defined as

$$S_{fi} = \psi_f^* O_g \psi_i \tag{1.7}$$

where S_{fi} is the S-matrix amplitude for the emission of a graviton in scattering from state ψ_i to state ψ_f. Clearly O_g is not a functional of ψ. In order to show the relationship between S_{fi} and the classical quadrupole formula, we first make a non-relativistic reduction of the matrix elements as after all the quadrupole formula is expected to make sense only when the particle motions are non-relativistic. After the non-relativistic reduction, the quantum-mechanical complementarity between space–time and energy momentum descriptions is invoked to find the space–time form of O_g. It will be seen that O_g is exactly the third time derivative of the quadrupole moment for the material distribution. This way one need not calculate such complicated quantities as total cross sections, frequency distribution, etc, to arrive at the classical limit.

We extract the one-graviton transition operator for the situation where the unequal-mass particles scatter gravitationally both in the Born approximation as well as in the distorted-wave approximation. We show that one recovers the same transition operator (as one should, since O_g is not a functional of the scattering states). Next we repeat the calculations for the electromagnetic scattering case and recover the same operator, thus implying the universal applicability of the quadrupole formula for gravitational radiation emission when the accelerating mechanisms are gravitational and otherwise. On the basis of the remark that O_g is not a functional of ψ we argue that a consistent way of calculating the emission of gravitational radiation from bound systems is to use the O_g in (1.7) derived from scattering studies but replace ψ_f, ψ_i by the appropriate bound-state wavefunctions. For the classical calculations this means that one takes the space–time form of O_g derived from scattering studies and depending on whether one wishes to derive the energy loss during scattering or from a classically bound system one evaluates O_g for the particular classical trajectories desired. This is indeed the standard interpretation of the quadrupole formula.

So our calculational strategy is as follows: in § 2 we set up Feynman rules for the interaction of gravitational fields with neutral as well as electrically charged matter fields.

In § 3 we use the results of § 2 to calculate the amplitudes for one-graviton emission both for the gravitational scattering case as well as the electromagnetic scattering case. In the scattering calculations the problem of the absence of incoming radiation is handled trivially. The non-relativistic limit of these amplitudes is then taken.

In § 4 we discuss the problem of the modification of the asymptotic in (out) states owing to the long-range nature of gravitation. A new scheme for generating S-matrix elements around these modified states is presented. Conceptual difficulties and some ideas on how to improve these schemes in the future are discussed. Again the amplitude is calculated in the non-relativistic limit.

In § 5 we argue the way to obtain the one-graviton transition operator and then show how to recover the classical energy-loss formulae from it.

In § 6 we present our discussions and conclusions.

2. Feynman rules

Most of the contents of this section except for the result on the graviton propagator in arbitrary gauge may be found in many places in the existing literature (Duff 1975). First we discuss the coupling of neutral scalar matter fields to gravitational fields. The Lagrangian density describing neutral scalar matter fields is given by

$$\mathscr{L}^0_{\rm M} = -\tfrac{1}{2}(\phi_{,\mu}\eta^{\mu\nu}\phi_{,\nu} + m^2\phi^2) \tag{2.1}$$

where $\phi_{,\mu} = \partial_\mu\phi$ and $\eta^{\mu\nu}$ is the Minkowski metric. The simplest Lagrangian density describing the interaction of the gravitational field with the scalar field is obtained by replacing $\eta^{\mu\nu}$ in $\mathscr{L}^0_{\rm M}$ by $g^{\mu\nu}$, all derivatives by covariant derivatives with respect to $g^{\mu\nu}$ and multiplying by $\sqrt{-g}$:

$$\mathscr{L}_{\rm M} = -\tfrac{1}{2}\sqrt{-g}(\phi_{,\mu}\phi_{,\nu}g^{\mu\nu} + m^2\phi^2). \tag{2.2}$$

The Lagrangian for the gravitational field is dictated by the action for general relativity, namely

$$\mathscr{L}_g = \frac{1}{16\pi G}\sqrt{-g}R. \tag{2.3}$$

From the computational point of view it turns out to be useful to work in the Goldberg form with the redefinition $\sqrt{-g}g^{\mu\nu} = \tilde{g}^{\mu\nu}(\tilde{g} = g)$; then (2.3) takes the particularly simple form

$$\mathscr{L}_g = \frac{1}{\kappa^2}(2\tilde{g}^{\rho\sigma}\tilde{g}_{\lambda\mu}\tilde{g}_{\eta\nu} - \tilde{g}^{\rho\sigma}\tilde{g}_{\mu\eta}\tilde{g}_{\lambda\nu} - 4\delta^\sigma_\eta\delta^\rho_\lambda\tilde{g}_{\mu\nu})\cdot\tilde{g}^{\mu\eta}{}_{,\rho}\tilde{g}^{\lambda\nu}{}_{,\sigma} \tag{2.4}$$

with

$$\kappa^2 = 16\pi G.$$

The matter Lagrangian takes the form

$$\mathscr{L}_{\mathrm{M}} = -\tfrac{1}{2}(\phi_{,\mu}\phi_{,\nu}\tilde{g}^{\mu\nu} + \sqrt{-\tilde{g}}m^2\phi^2). \tag{2.5}$$

Likewise the Lagrangian describing a charged scalar field and the electromagnetic field interacting with the gravitational field is

$$\mathscr{L}_{\mathrm{ch}} = -\tfrac{1}{2}[(\phi_{,\mu} - \mathrm{i}eA_\mu\phi)\tilde{g}^{\mu\nu}(\phi^*_{,\nu} + \mathrm{i}eA_\nu\phi^*) + \sqrt{-\tilde{g}}m^2\phi^*\phi] - \tfrac{1}{4}(-\tilde{g})^{-1/2}F_{\mu\nu}F_{\lambda\kappa}\tilde{g}^{\nu\kappa}\tilde{g}^{\mu\lambda}. \tag{2.6}$$

2.1. Perturbation theory

The gravitational field $\tilde{h}^{\mu\nu}$ is identified as

$$\tilde{g}^{\mu\nu} = \eta^{\mu\nu} + 2\kappa\tilde{h}^{\mu\nu} \qquad g^{\mu\nu} = \eta^{\mu\nu} + 2\kappa h^{\mu\nu}. \tag{2.7}$$

With this definition one obtains the conventional normalisation of one particle per unit volume. From (2.7) we get

$$\tilde{h}^{\mu\nu} = h^{\mu\nu} - \tfrac{1}{2}h^\alpha_\alpha\eta^{\mu\nu} + \mathrm{O}(h^2). \tag{2.8}$$

The functional derivative of $\mathscr{L}_{\mathrm{M}}$ in (2.2) with respect to $h^{\mu\nu}$ yields the source $\theta_{\mu\nu}$ of the field $h^{\mu\nu}$

$$\theta_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} + \eta_{\mu\nu}\mathscr{L}_{\mathrm{M}} + \mathrm{O}(h) \tag{2.9}$$

which satisfies

$$\partial^\mu\theta_{\mu\nu} = \mathrm{O}(h). \tag{2.10}$$

The Lagrangian $\mathscr{L}_{\mathrm{M}}$ can be rewritten as

$$\mathscr{L}_{\mathrm{M}} = \mathscr{L}^0_{\mathrm{M}} + \kappa h^{\mu\nu}\theta_{\mu\nu}. \tag{2.11}$$

By virtue of (2.10), any modification of $h^{\mu\nu}$ of the type

$$h^{\mu\nu} \to h^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu \tag{2.12}$$

leaves the action $\int\mathscr{L}_{\mathrm{M}}\,\mathrm{d}^4x$ invariant. The bilinear terms in $h^{\mu\nu}$ of $\mathscr{L}_{\mathrm{g}}$ are likewise invariant under (2.12). Invariance under (2.12) is the statement of the gravitational gauge invariance. For the source of the Goldberg field we obtain

$$\partial^\mu\tilde{\theta}_{\mu\nu} = \tfrac{1}{2}\partial_\nu\tilde{\theta}^\alpha_\alpha + \mathrm{O}(h). \tag{2.13}$$

When we do not drop the $\mathrm{O}(h)$ terms in $\theta_{\mu\nu}$, it is no longer conserved. But then $\theta_{\mu\nu}$ is not the entire source of $h^{\mu\nu}$ as the cubic terms in $\mathscr{L}_{\mathrm{g}}$ generate additional source terms. These terms yield a $t_{\mu\nu}(h)$ which together with $\theta_{\mu\nu}$ (inclusive of $\mathrm{O}(h)$ terms in it) is conserved by virtue of the field equations.

Now far away from the sources

$$h^{\mu\nu} \to h^{\mu\nu}_{\mathrm{rad}}$$

and it is a very legitimate approximation to ignore terms $\mathrm{O}(h^2_{\mathrm{rad}})$ and the source of $h^{\mu\nu}_{\mathrm{rad}}$ is $\theta^{\mathrm{rad}}_{\mu\nu}$ which is conserved; this means, as argued before, that h_{rad} has an arbitrariness in it which can be expressed as

$$h^{\mu\nu}_{\mathrm{rad}} \to h^{\mu\nu}_{\mathrm{rad}} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu. \tag{2.14}$$

In momentum-space language, this means that the polarisation tensor $\varepsilon^{\mu\nu}(k)$ of the emitted graviton is arbitrary up to terms of the type $k^\mu\xi^\nu(k)+k^\nu\xi^\mu(k)$ for arbitrary $\xi(k)$.

The S-matrix element is of the form

$$S=\varepsilon^{\mu\nu}(k)M_{\mu\nu} \tag{2.15}$$

and under the change $\varepsilon^{\mu\nu}(k)\to\varepsilon^{\mu\nu}(k)+k^\mu\xi^\nu(k)+k^\nu\xi^\mu(k)$ S should remain unchanged. This means

$$\delta S=\xi^\nu k^\mu M_{\mu\nu}+\xi^\mu k^\nu M_{\mu\nu}=0 \qquad \text{for all } \xi. \tag{2.16}$$

In other words

$$k^\mu M_{\mu\nu}=0. \tag{2.17}$$

This is the statement of gravitational gauge invariance for the S-matrix elements. When calculations are performed in the Goldberg form (2.21) becomes

$$k^\mu \tilde{M}_{\mu\nu}=\tfrac{1}{2}k_\nu\tilde{M}^\alpha_\alpha. \tag{2.18}$$

The arbitrariness in the choice of $\varepsilon^{\mu\nu}(k)$ allows one to impose on $\varepsilon^{\mu\nu}(k)$ the conditions

$$k_\mu\varepsilon^{\mu\nu}=0 \qquad \varepsilon^\alpha_\alpha=0 \qquad \varepsilon^\mu_0=0. \tag{2.19}$$

The first of these is dictated by the wave equation satisfied by $h^{\mu\nu}$; the remaining conditions can be imposed by a proper choice of $\xi(k)$. There are eight independent conditions implied by (2.19). This means that of the ten components of the polarisation tensor $\varepsilon^{\mu\nu}$ only *two* are physically independent degrees of freedom and these are the two transverse degrees of freedom of gravitational radiation.

By extracting the quadratic terms in $h^{\mu\nu}$ in $\mathscr{L}_g$, we find the equation satisfied by the free field is

$$-\partial^2 h_{\mu\nu}+\partial_\mu\partial^\lambda h_{\lambda\nu}+\partial_\nu\partial^\lambda h_{\lambda\mu}-\partial_\mu\partial_\nu h^\alpha_\alpha=0. \tag{2.20}$$

Just as in electrodynamics it is not possible to invert this to yield the Green functions. To this purpose we add a gauge fixing term to $\mathscr{L}_g$:

$$\mathscr{L}_g^{GB}=\frac{1}{2\lambda}\tilde{h}^{\mu\alpha}{}_{,\mu}\tilde{h}^{\nu\beta}{}_{,\nu}\eta_{\alpha\beta}. \tag{2.21}$$

The propagator is then calculated to be

$$\begin{aligned}D_{+\mu\nu,\alpha\beta}=\frac{i}{2k^2}\bigg((\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta})+\frac{1-\lambda}{2\lambda-1}\eta_{\mu\nu}\eta_{\alpha\beta}\\
+\frac{2(1-\lambda)}{2\lambda-1}\frac{1}{k^2}(k_\mu k_\nu\eta_{\alpha\beta}+k_\alpha k_\beta\eta_{\mu\nu})\\
+\frac{\lambda-1}{2\lambda-1}\frac{1}{k^2}(k_\mu k_\alpha\eta_{\nu\beta}+k_\mu k_\beta\eta_{\nu\alpha}+k_\nu k_\beta\eta_{\mu\alpha}+k_\nu k_\alpha\eta_{\mu\beta})\bigg).\end{aligned} \tag{2.22}$$

The normalisation factor $\frac{1}{2}$ should be noted. Clearly $\lambda=1$ produces a very simple form of the propagator. This is equivalent to the Lorentz gauge in electrodynamics and corresponds to the De Donder choice of coordinates in general relativity. In a similar fashion the photon propagator in arbitrary gauge is

$$D_{+\mu\nu}=\frac{\mathrm{i}}{k^2}\left(\eta^{\mu\nu}+(\alpha-1)\frac{k^\mu k^\nu}{k^2}\right). \tag{2.23}$$

2.2. The vertices

Now we collect the various vertices that are required for our calculations. First we give the results coming from the Lagrangians (2.4) and (2.5) using (2.7): these vertices are displayed in figure 1.

1(a) $$\Gamma_{\mu\nu}(p',p)=-\mathrm{i}\kappa(p_\mu p'_\nu+p_\nu p'_\mu+\eta_{\mu\nu}m^2) \tag{2.24a}$$

1(b) $$\Gamma_{\mu\nu,\alpha\beta}(p',p:k_1,k_2)=-\mathrm{i}\kappa^2 m^2(\eta_{\alpha\beta}\eta_{\mu\nu}-\eta_{\alpha\mu}\eta_{\beta\nu}-\eta_{\alpha\nu}\eta_{\beta\mu}). \tag{2.24b}$$

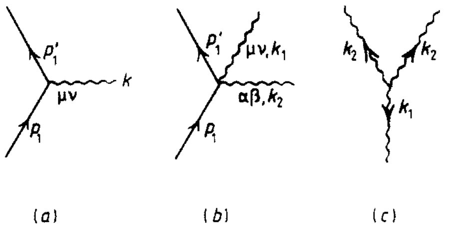

Figure 1.

The three-graviton vertex is really very complicated. We denote it as

1(c) $\Gamma_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3}(k_1,k_2,k_3)$ with $k_1+k_2+k_3=0$.

We introduce the symbolic notation $(\alpha\beta)\equiv a$, etc. In this notation we can write the three-graviton vertex as

$$\Gamma_{a_1a_2a_3}=-\tfrac{1}{4}\mathrm{i}\kappa(\eta^{a_1a_3a_2;\rho\sigma}k^3_\rho k^2_\sigma+\eta^{a_1a_2a_3;\rho\sigma}k^2_\rho k^3_\sigma+\eta^{a_2a_3a_1;\rho\sigma}k^3_\rho k^1_\sigma+\eta^{a_2a_1a_3;\rho\sigma}k^1_\rho k^3_\sigma$$
$$+\eta^{a_3a_2a_1;\rho\sigma}k^2_\rho k^1_\sigma+\eta^{a_3a_1a_2;\rho\sigma}k^1_\rho k^2_\sigma) \tag{2.24c}$$

where

$$\eta^{abc;\rho\sigma}\equiv\eta^{\alpha\beta,\mu\nu,\lambda\kappa;\rho\sigma}$$
$$=4(\eta^{\mu\alpha}\eta^{\beta\kappa}\eta^{\lambda\rho}\eta^{\nu\sigma})+2\eta^{\alpha\rho}\eta^{\beta\sigma}\eta^{\mu\lambda}\eta^{\nu\kappa}-4\eta^{\beta\mu}\eta^{\rho\sigma}\eta^{\nu\kappa}\eta^{\lambda\alpha}$$
$$-\eta^{\rho\alpha}\eta^{\sigma\beta}\eta^{\mu\nu}\eta^{\lambda\kappa}+2\eta^{\rho\sigma}\eta^{\lambda\kappa}\eta^{\mu\alpha}\eta^{\beta\nu}.$$

The Bose symmetry of the three-graviton vertex is very transparent as written in (2.24c).

Now we give the vertices that are generated by the Lagrangian (2.6). These are shown in figure 2. The broken lines are photons and the wavy lines are gravitons. The vertex 2(a) is the same as 1(a) and we have

2(b) $$\Gamma_\mu(p,p')=\mathrm{i}e(p+p')_\mu \tag{2.25b}$$

2(c) $$\Gamma_{\mu\nu,\alpha}(p,p';k,q)=\tfrac{1}{2}\mathrm{i}e\kappa[(p+p')_\mu\eta_{\alpha\nu}+(p+p')_\nu\eta_{\alpha\mu}] \tag{2.25c}$$

2(d) $$\Gamma_{\mu\nu,\alpha,\beta}(k,q,q')=\mathrm{i}\kappa[(q'_\mu q_\nu+q'_\nu q_\mu)\eta_{\alpha\beta}+q\cdot q'(\eta_{\mu\beta}\eta_{\nu\alpha}+\eta_{\mu\alpha}\eta_{\nu\beta})$$
$$-(q'_\alpha q_\nu\eta_{\mu\beta}+q'_\alpha q_\mu\eta_{\nu\beta}+q'_\mu q_\beta\eta_{\nu\alpha}+q'_\nu q_\beta\eta_{\mu\alpha})]. \tag{2.25d}$$

With these Feynman rules we are ready to proceed with our calculations.

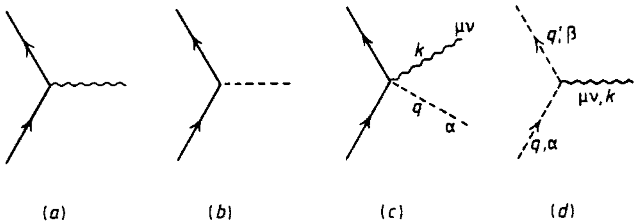

Figure 2.

3. Born approximation amplitudes

3.1. The case of gravitational scattering

The diagrams that contribute to the one-graviton emission amplitude are shown in figure 3. We classify these diagrams as follows: diagrams (*a*)–(*d*) only involve vertices 1(*a*); for this reason we shall call this set 'one-graviton graphs'. Likewise, (*e*) and (*f*) are called 'two-graviton graphs' and (*g*) is called the 'three-graviton graph'.

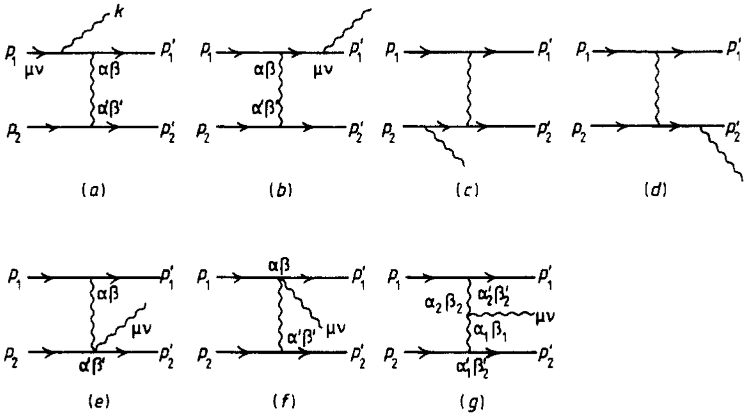

Figure 3.

Even though the amplitudes are easily written down, they are algebraically very tedious. Hence we shall only state various properties of these amplitudes without writing them down explicitly. The amplitudes are independent of λ and α, signifying their gravitational and electromagnetic gauge invariance as far as the non-radiative fields are concerned. Further, (2.18) has been verified to hold for all the amplitudes, signifying gauge invariance of the radiated field. These calculations also serve to check the correctness of various manipulations.

3.1.1. Non-relativistic reduction. We specialise to the centre-of-mass frame of the incoming particles m_1, m_2; we define

$$p_1=(E_1, \tfrac{1}{2}\boldsymbol{p}) \qquad p_2=(E_2, -\tfrac{1}{2}\boldsymbol{p}). \tag{3.1}$$

Likewise we denote

$$p_1'=(E_1', \tfrac{1}{2}\boldsymbol{p}') \quad \text{and} \quad p_2'=(E_2', -\tfrac{1}{2}\boldsymbol{p}'). \tag{3.2}$$

If we call $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ the position vectors of particles 1 and 2, we have

$$M\boldsymbol{R} = m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 \qquad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \tag{3.3}$$

here $\boldsymbol{R}$, $\boldsymbol{r}$ are the centre-of-mass ordinates and relative separation; M is the total mass of the system

$$m_1\dot{\boldsymbol{r}}_1 + m_2\dot{\boldsymbol{r}}_2 = M\dot{\boldsymbol{R}} = \boldsymbol{p}_1' + \boldsymbol{p}_2' = -\boldsymbol{k} \tag{3.4}$$

and since $\boldsymbol{p}_1' + \boldsymbol{p}_2' + \boldsymbol{k} = 0$, therefore

$$m_1\dot{\boldsymbol{r}}_1 - m_2\dot{\boldsymbol{r}}_2 = (m_1 - m_2)\dot{\boldsymbol{R}} + 2\mu\dot{\boldsymbol{r}}; \tag{3.5}$$

hence

$$\boldsymbol{p}_1' - \boldsymbol{p}_2' = \boldsymbol{p}' - \frac{m_1 - m_2}{M}\boldsymbol{k} \qquad \text{where } \boldsymbol{p}' = 2\mu\dot{\boldsymbol{r}} \tag{3.6}$$

and

$$\boldsymbol{p}_1' = \tfrac{1}{2}\boldsymbol{p}' - \frac{m_1}{M}\boldsymbol{k} \qquad \boldsymbol{p}_2' = -\tfrac{1}{2}\boldsymbol{p}' - \frac{m_2}{M}\boldsymbol{k}. \tag{3.7}$$

Thus the total final energy is

$$E_f = m_1 + m_2 + k^0 + \frac{p'^2}{8\mu} + \frac{k^2}{8M}.$$

The initial energy is

$$E_i = m_1 + m_2 + p^2/8\mu \tag{3.8}$$

since $k^0 \ll M$, $k^0 \sim (p^2 - p'^2)/8\mu$ and

$$|\boldsymbol{k}| \sim \frac{(p+p')(p-p')}{8\mu} \ll |p|, |p'|. \tag{3.9}$$

In consequence we can write

$$\boldsymbol{p}_1' = \tfrac{1}{2}\boldsymbol{p}' \qquad \boldsymbol{p}_2' = -\tfrac{1}{2}\boldsymbol{p}'. \tag{3.10}$$

Hence the final four-momenta are

$$p_1' = (E_1', \tfrac{1}{2}\boldsymbol{p}') \qquad p_2' = (E_2', -\tfrac{1}{2}\boldsymbol{p}') \tag{3.11}$$

$$k^0 \sim (v/c)(p, p'). \tag{3.12}$$

Now we are ready to work out the non-relativistic limit of the amplitude to emit a graviton.

Because of the gauge conditions (2.19) we can neglect all terms proportional to $\eta^{\mu\nu}$ in the amplitude. We skip the arithmetical details and only indicate the final results: (the three-graviton graph contribution)

$$M_{ij}^3 \approx 4\mathrm{i}\kappa^3 \frac{q_i q_j}{(q^2)^2} m_1^2 m_2^2 \tag{3.13}$$

where

$$\boldsymbol{q} = \tfrac{1}{2}(\boldsymbol{p} - \boldsymbol{p}'). \tag{3.14}$$

Thus M^3 is of order $\frac{1}{2}\kappa^3 m^4$; by straightforward evaluation we find (two-graviton graphs)

$$M_{ij}^2 \sim O(\kappa^3 m^2 q^2). \tag{3.15}$$

Thus the two-graviton graph contributions are q^2/m^2 reduced relative to M^3 and hence can be neglected. The structure of the one-graviton terms is more complicated and, after a little algebra, we find

$$M'_{ij} \approx \tfrac{1}{2}\mathrm{i}\kappa^3 \frac{m_1 m_2}{\omega} \frac{M}{q^2}(p^i p^j - p'^i p'^j). \tag{3.16}$$

Thus the total amplitude in the non-relativistic limit is

$$M_{ij} \approx 4\mathrm{i}\kappa^3 \frac{m_1 m_2}{q^2}\left(\frac{M}{8\omega}(p^i p^j - p'^i p'^j) + \mu M \frac{q^i q^j}{q^2}\right) \tag{3.17}$$

where $\mu = m_1 m_2/M$ and $M = m_1 + m_2$.

3.2. *The case of electromagnetic scattering*

The diagrams that contribute to the amplitude for emission of gravitons during electromagnetic scattering are shown in figure 4. The kinematic structure of invariants is exactly as for gravitational scattering. The diagrams in figure 4 are evaluated using the vertices (2.24*a*), (2.25*b*), (2.25*c*) and (2.25*d*). The photon propagator as given in (2.23) is needed also.

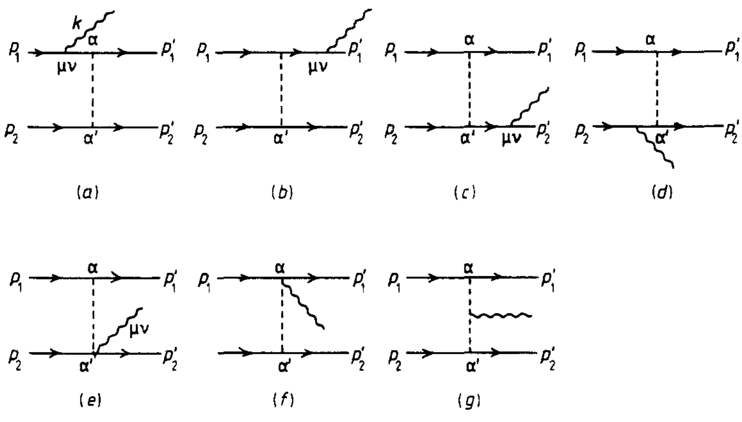

Figure 4.

The calculations proceed exactly as for gravitational scattering. The significant results are that the amplitude is independent of the photon gauge fixing parameter and the total amplitude satisfies the gravitational gauge invariance condition (2.18). The non-relativistic reduction is done exactly as for gravitational scattering and since this is the only result we are interested in we shall quote only the final result of our non-relativistic analysis:

$$M_{ij}^{e} = -8\mathrm{i}\kappa \frac{(m_1 + m_2)}{q^2} e_1 e_2 \left(\frac{p_i p_j - p'_i p'_j}{8\omega} + \mu \frac{q_i q_j}{q^2}\right). \tag{3.18}$$

We recognise that the amplitudes (3.17) and (3.18) are simply proportional to each other. Incidentally, the charges in (3.18) are measured in rationalised units.

4. The modified Born approximation

As mentioned in the introduction, the long-range nature of the gravitational interactions makes the Born approximation based on incoming and outgoing plane-wave states somewhat unreliable. This problem exists in electrodynamics also. In the case of electrodynamics a fully relativistic treatment of the necessary modifications can be carried out. In the gravitational case, however, such a fully relativistic treatment is very difficult. But for the problem of our interest where only non-relativistic motions are involved the situation is tractable.

In the Born approximation scheme employed in § 3, the initial and final states are plane-wave states; in particular the initial state is a two-particle state where each particle is taken to be in a plane-wave state. The state of the two-particle system is then described by the product wavefunction

$$\begin{aligned}\chi(\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2) &= \psi_1(\boldsymbol{r}_1, t_1)\psi_2(\boldsymbol{r}_2, t_2)\\ &= \exp(\mathrm{i}E_1 t_1 - \mathrm{i}\boldsymbol{p}_1 \cdot \boldsymbol{r}_1)\exp(\mathrm{i}E_2 t_2 - \mathrm{i}\boldsymbol{p}_2 \cdot \boldsymbol{r}_2)\end{aligned} \tag{4.1}$$

in the 'many-time' formalism. When the motion of the particles is non-relativistic one only needs (4.1) for $t_1 \sim t_2$ and thus a 'single-time' formalism for the composite system may be used. In such a case

$$\chi(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = \exp[\mathrm{i}(E_1 + E_2)t]\exp(-\mathrm{i}\boldsymbol{p}_1 \cdot \boldsymbol{r}_1 - \mathrm{i}\boldsymbol{p}_2 \cdot \boldsymbol{r}_2). \tag{4.2}$$

Introducing the notations of equations (3.3)–(3.6) we find

$$E_1 + E_2 = P^2/2M + p^2/8\mu$$

and

$$\boldsymbol{p}_1 \cdot \boldsymbol{r}_1 + \boldsymbol{p}_2 \cdot \boldsymbol{r}_2 = \boldsymbol{P} \cdot \boldsymbol{R} + \tfrac{1}{2}\boldsymbol{p} \cdot \boldsymbol{r}. \tag{4.3}$$

Hence

$$\begin{aligned}\chi(\boldsymbol{r}_1, \boldsymbol{r}_2, t) &= \chi(\boldsymbol{R}, \boldsymbol{r}, t)\\ &= \exp[\mathrm{i}(P^2/2M + p^2/8\mu)t]\exp(-\mathrm{i}\boldsymbol{P} \cdot \boldsymbol{R})\exp(-\tfrac{1}{2}\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{r}).\end{aligned} \tag{4.4}$$

Thus the centre-of-mass and relative coordinate motions decouple. When there are long-range forces present the composite wavefunction $\chi(r_1, r_2, t)$ can no longer be factored into the single-particle functions $\exp(\mathrm{i}E_1 t - \mathrm{i}\boldsymbol{p}_1 \cdot \boldsymbol{r}_1)$, etc. This is because even at very large distances some effect of the long-range potential persists. However, the separation into the centre-of-mass and relative coordinates still holds. But now instead of the simple $\exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{r})$ factor one has a much more complicated dependence on $\boldsymbol{p}$ and $\boldsymbol{r}$.

Now we make the simplifying assumption that all the long-range modifications are entirely produced by the $1/r$ part of the potential. This assumption is justified by the fact that in describing the propagation of perturbations in a Schwarzchild background, the asymptotic form of the wavefunction is precisely that of a Coulomb distorted plane wave. In such a situation the composite wavefunction takes the form

$$\chi(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = N \exp[\mathrm{i}(P^2/2M + p^2/8\mu)t] \times \exp(-\mathrm{i}\boldsymbol{P}\cdot\boldsymbol{R}) \exp(-\tfrac{1}{2}\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}) F(\mathrm{i}\nu, 1, \tfrac{1}{2}\mathrm{i}pr - \tfrac{1}{2}\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}). \tag{4.5}$$

In (4.5) F is the confluent hypergeometric function and N is a normalisation factor. For the gravitational problem $\nu = Gm_1m_2\mu/p$ ($\hbar = c = 1$) and in general ν can be very large, thus invalidating a perturbative expansion in G; we claim here that all the non-perturbative aspects of the S-matrix elements are contained in (4.5). It is clear that equation (4.5) cannot be separated into factors depending on $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ separately. We can, however, expand the r dependence of equation (4.5) in the plane-wave basis $\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r})$, i.e.

$$\chi(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = N \exp[\mathrm{i}(P^2/2M + p^2/8\mu)t] \exp(-\mathrm{i}\boldsymbol{P}\cdot\boldsymbol{R}) \times \int \mathrm{d}\boldsymbol{q} \exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}) \psi(\mathrm{i}\nu, \boldsymbol{p}, \boldsymbol{q}). \tag{4.6}$$

Thus we can interpret the long-range modified composite wavefunction, equation (4.5), as a wavepacket built out of the plane-wave states suitable for the Born approximation. Now we can write a product decomposition for

$$\exp(-\mathrm{i}\boldsymbol{P}\cdot\boldsymbol{R}) \exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}) = \exp(-\mathrm{i}\boldsymbol{q}_1\cdot\boldsymbol{r}_1) \exp(-\mathrm{i}\boldsymbol{q}_2\cdot\boldsymbol{r}_2) \tag{4.7}$$

where

$$\boldsymbol{q}_1 = \frac{m_1}{M}\boldsymbol{P} + \boldsymbol{q} \qquad \boldsymbol{q}_2 = \frac{m_2}{M}\boldsymbol{P} - \boldsymbol{q}.$$

The t integrations at various vertices give the energy conservation δ function while $\boldsymbol{R}$ integrations give the momentum conservation δ functions; clearly there is no conservation of the relative momenta $\boldsymbol{p}$, etc.

Now the modified S-matrix elements are obtained from the Lagrangians of (2.4), (2.5) and (2.6) by expanding the field variables not in a plane-wave basis but in the basis provided by (4.5). Clearly a single field has no consistent expansion; only product fields $\phi_1(x)\phi_2(y)$ can be expanded in the basis provided by (4.5). One may anticipate some problems coming from the derivative terms. It turns out that as long as only spatial derivatives are involved, a consistent scheme can be derived for obtaining the S-matrix elements in terms of the basis of (4.5). Both in electromagnetic and gravitational radiation problems, the freedom in the choice of gauge ensures that it is possible to keep only spatial derivatives in the action integrals. The whole procedure boils down to using (4.6) and treating the problem as if it were a Born approximation performed with the plane-wave states $(E_1, \boldsymbol{q}_1)$ and $(E_2, \boldsymbol{q}_2)$; a minor subtlety arises due to the fact that only $E_1^2 - \boldsymbol{p}_1^2 = m_1^2$ and not $E_1^2 - \boldsymbol{q}_1^2$ but the Fourier transform $\psi(\mathrm{i}\nu, \boldsymbol{p}, \boldsymbol{q})$ of the confluent hypergeometric function is of the form

$$\psi(\mathrm{i}\nu, \boldsymbol{p}, \boldsymbol{q}) = \frac{8\pi\nu p}{(p^2 - q^2)^{1+\mathrm{i}\nu}} \frac{1}{|\boldsymbol{p} - \boldsymbol{q}|^{2-2\mathrm{i}\nu}} \tag{4.8}$$

which indicates that the important values of $\boldsymbol{q}$ are such that they lie very close to $\boldsymbol{p}$ and thus the errors introduced by treating the particles as if they were on the mass shell are negligibly small. The rest of the analysis proceeds more or less as in § 3 and the final result in the non-relativistic limit is

$$M_{ij} = 4i\kappa^3 m_1 m_2 \int \mathrm{d}\boldsymbol{q}\, \mathrm{d}\boldsymbol{q}'\, \psi_-^*(-i\nu', \boldsymbol{p}', \boldsymbol{q}')\psi_+(i\nu, \boldsymbol{p}, \boldsymbol{q})$$

$$\times\left(\mu M\frac{\tilde{q}_i\tilde{q}_j}{(\tilde{q}^2)^2} + \frac{M}{8\omega\tilde{q}^2}(q^i q^j - q'^i q'^j)\right) \tag{4.9}$$

where $\tilde{q} = q - q'$; the Fourier transforms carry the subscript $\pm$ to indicate that the Coulomb modifications of the initial- and final-state plane waves are different. The initial state is that modification whose asymptotic form consists of a plane wave and an *outgoing* spherical wave; likewise the final state is such that its asymptotic form is a plane wave and an *ingoing* spherical wave.

When ν and ν' are small, we can use

$$\psi_\pm(i\nu, \boldsymbol{p}, \boldsymbol{q}) \xrightarrow[\nu\to 0]{} \delta^3(\boldsymbol{p} - \boldsymbol{q}) \tag{4.10}$$

and verify that (4.9) reduces to the Born approximation results.

The double-pole term in (4.9) comes from 'three-graviton' graphs and the single-pole term comes from one-graviton graphs. In general a precise evaluation of M_{ij} in (4.9) is very hard to perform. But we shall show in the next section that this explicit evaluation is really not warranted to arrive at the classical limit.

In deriving (4.9) we treated the graviton wavefunction as a plane wave. Strictly speaking this is not correct as the graviton also carries 'gravitational charge' and its wavefunction should be modified by appropriate long-range effects; we again assume that the $1/r$ part of the gravitational interaction is entirely responsible for this long-range influence. The justification for this is again derived on the basis of the equations for tensor perturbations around a Schwarzchild metric (Visveshwara, private communication); these equations indicate that the asymptotic modifications are the same irrespective of whether the perturbations are scalar, vector or tensorial. The graviton wavefunction therefore becomes

$$h_{\mu\nu} = \varepsilon_{\mu\nu}(k)\exp(i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r})F_-(-2i\alpha, 1, ikr + i\boldsymbol{k}\cdot\boldsymbol{r}) \tag{4.11}$$

where $\alpha = GM\omega/c^3$.

The factor 2 is associated with α because relativistic particles 'fall' at twice the rate for non-relativistic particles. The asymptotic behaviour of equation (4.11) is

$$h_{\mu\nu} \sim \varepsilon_{\mu\nu}(k)\exp[i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r} + (2iGM\omega/c^3)\ln r]. \tag{4.12}$$

Thus we see that on using (4.11) instead of the plane wavefunctions for the gravitons, we have completely accounted for the fact that the 'true light cones' deviate logarithmically from the 'flat space light cones' at large distances. This is one of the criticisms of Ehlers *et al* of the existing methods of approximation. It is to be noted that by defining a new time variable

$$t^* = t + (2GM/c^3)\ln r \tag{4.13}$$

the wavefunction of (4.11) can be made to look like a plane-wave solution:

$$h_{\mu\nu} \sim \varepsilon_{\mu\nu}(k)\exp(i\omega t^* - i\boldsymbol{k}\cdot\boldsymbol{r}). \tag{4.14}$$

The variable t^* is precisely the one that has been introduced by Anderson (1980) to alleviate the problem that asymptotically the true light cones differ logarithmically from the flat space ones. Now we come to the question: what then is the influence of (4.11) on the result (4.9)?

A not so rigorous resolution is as follows. for weakly bound systems, ω is expected to be of the order $\sqrt{GM/R^3}$ where R is the typical size of the bound system. This means that $GM\omega/R^3 \sim (GM/Rc^2)^{3/2}$, which is very small for weakly bound systems. The consequence of the smallness of α in (4.11) is that one may safely use the plane-wave states in place of (4.11) and the result (4.9) will be reliable.

Actually, a more rigorous argument is available. Just as we Fourier decomposed the matter wavefunctions as in (4.6), we can likewise Fourier decompose the graviton wavefunction (4.11); then equation (4.9) is modified as follows:

$$\tilde{M}_{ij} = 4\mathrm{i}\kappa^3 m_1 m_2 \int \mathrm{d}\boldsymbol{q}\, \mathrm{d}\boldsymbol{q}'\, \mathrm{d}\boldsymbol{k}'\, \psi_-^*(-\mathrm{i}\nu', \boldsymbol{p}', \boldsymbol{q}')\psi_-^*(-2\mathrm{i}\alpha, \boldsymbol{k}, \boldsymbol{k}')\psi_+(\mathrm{i}\nu, \boldsymbol{p}, \boldsymbol{q})$$

$$\times\left(\mu M \frac{\tilde{q}_i\tilde{q}_j}{(\tilde{q}^2)^2} + \frac{M}{8\omega\tilde{q}^2}(q_iq_j - q_i'q_j')\right). \tag{4.15}$$

For the non-relativistic system considered here, the crucial point is that the terms within the brackets *are independent of* k'! This means that the $\mathrm{d}k'$ integration in (4.15) may trivially be carried out on noting

$$\int \mathrm{d}\boldsymbol{k}'\, \psi_-^*(-2\mathrm{i}\alpha, \boldsymbol{k}, \boldsymbol{k}') = 1 \qquad \text{for all } \alpha. \tag{4.16}$$

Thus (4.15) reduces exactly to (4.9) even though we have now made a rigorous inclusion of the deviation of the true light cones from the flat space ones. We again emphasise that this miraculous simplification disappears the moment we consider relativistic particle motions; but no one expects the quadrupole formula to be valid in those circumstances.

5. The one-graviton transition operator

We first demonstrate how to obtain the one-graviton transition operator from our Born approximation amplitudes and then repeat the calculation for the amplitudes obtained in § 4. For the Born amplitudes, we combine the results of (3.17) and (3.18) and arrive at the amplitude to emit a graviton during the scattering of two particles of masses and charges (m_1, e_1) and (m_2, e_2) respectively:

$$M_{ij} = \frac{-\mathrm{i}}{\omega} M 2\kappa(4e_1e_2 - 2\kappa^2 m_1 m_2)\left(\mu\omega \frac{q_iq_j}{(q^2)^2} + \frac{p_ip_j - p_i'p_j'}{8q^2}\right) \tag{5.1}$$

$$\boldsymbol{q} = \tfrac{1}{2}(\boldsymbol{p} - \boldsymbol{p}') \qquad \omega = (p^2 - p'^2)/8\mu.$$

It is interesting to note that when $2e_1e_2 = \kappa^2 m_1 m_2$, the amplitude to emit radiation (gravitational) vanishes in the approximation considered here.

Using $\frac{1}{2}\boldsymbol{p}' = \frac{1}{2}\boldsymbol{p} + \boldsymbol{q}$ we rewrite the expression within the brackets of (5.1) as follows:

$$\mu\omega \frac{q_iq_j}{(q^2)^2} - \frac{1}{4}\left(\frac{p_iq_j}{q^2} + \frac{q_jq_i}{q^2}\right) - \frac{1}{2}\frac{q_iq_j}{q^2}. \tag{5.2}$$

The initial-state wavefunction is of the form (see equation (4.4))

$$\psi_i = \exp[\mathrm{i}(P^2/2M + p^2/8\mu)t]\exp(-\mathrm{i}\boldsymbol{P}\cdot\boldsymbol{R})\exp(-\tfrac{1}{2}\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}). \tag{5.3}$$

Since the centre-of-mass variables only reproduce the overall momentum conservation δ functions we shall ignore them and write

$$\psi_i = \exp(\mathrm{i}p^2 t/8\mu - \tfrac{1}{2}\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}). \tag{5.4}$$

Likewise the final-state wavefunction is

$$\psi_f = \exp(\mathrm{i}p'^2 t/8\mu)\exp(-\tfrac{1}{2}\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{r}). \tag{5.5}$$

Now it is known very well that $1/q^2$ is the Fourier transform of the Coulomb potential, i.e.

$$\frac{1}{\boldsymbol{q}^2} = \int \mathrm{d}\boldsymbol{r}\frac{\exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r})}{4\pi r} \tag{5.6}$$

$$\frac{q^i}{\boldsymbol{q}^2} = \int \mathrm{d}\boldsymbol{r}\exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r})\left(-\mathrm{i}\nabla^i\frac{1}{4\pi r}\right). \tag{5.7}$$

Similarly we have

$$\frac{q^iq^j}{(\boldsymbol{q}^2)^2} = \int \mathrm{d}\boldsymbol{r}\exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r})\frac{1}{8\pi}\left(\frac{r_ir_j}{r^3} - \frac{\delta_{ij}}{r}\right) \tag{5.8}$$

and finally

$$\frac{q^iq^j}{\boldsymbol{q}^2} = \int \mathrm{d}\boldsymbol{r}\exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r})\left[\delta_{ij}\frac{\delta^3(r)}{3} + \left(\frac{\delta_{ij}}{r^3} - \frac{3r_ir_j}{r^5}\right)\right]. \tag{5.9}$$

Comparing (5.9) and (5.8), we see that the q_iq_j/q^2 term in (5.2) can be neglected in the classical limit. Finally, we recognize that the space–time equivalent of ω is the $\mathrm{i}\partial/\partial t$ operator; combining all these results we find

$$M_{ij} = -\frac{\mathrm{i}M\mu}{\omega}2\kappa(4e_1e_2 - 2\kappa^2 m_1m_2)\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\omega t}\psi_f X_{ij}\psi_i \tag{5.10}$$

where

$$X_{ij} = \frac{\mathrm{i}}{8\pi}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{r_ir_j}{r^3} + \frac{v_ir_j + v_jr_i}{r^3}\right). \tag{5.11}$$

We have used $\frac{1}{2}\boldsymbol{p} = \mu\,\mathrm{d}\boldsymbol{r}/\mathrm{d}t = \mu\boldsymbol{v}$ in deriving (5.11) (see equation (3.6)).

The structure X_{ij} as it stands in (5.11) is not very transparent. But on using the equations of motion

$$\mu\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{(4e_1e_2 - 2\kappa^2 m_1m_2)}{16\pi}\nabla_i\frac{1}{r} \tag{5.12}$$

one recognises that M_{ij} is precisely equal to

$$M_{ij} = -\frac{4m_1m_2}{\omega}\tfrac{1}{2}\kappa(\dddot{D}'_{ij})_{fi} \qquad D'_{ij} = \mu r_ir_j. \tag{5.13}$$

The proof of this statement is as follows:

$$\begin{aligned}\mu(r_ir_j)^{\dddot{}} &= \mu(r_iv_j + v_ir_j)^{\ddot{}}\\ &= \mu(2v_iv_j + r_i\dot{v}_j + \dot{r}_jv_i)^{\dot{}}\\ &= 2(\mu v_iv_j + 2\xi r_ir_j/r^3)^{\dot{}}\end{aligned}$$

using (5.12) and abbreviating $\frac{1}{16}(4e_1e_2 - 2\kappa^2 m_1 m_2)$ by ξ. Therefore

$$\mu(r_i r_j)^{\cdots} = 2(\mu v_i v_j + \xi r_i r_j / r^2)^{\cdot}$$
$$= 2\xi\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{r_i r_j}{r^3} + \frac{v_i r_j + v_j r_i}{r^3}\right). \qquad (5.14)$$

Including the polarisation tensor the amplitude to emit the graviton is

$$M = M_{ij}\varepsilon^{ij} = -\frac{4m_1m_2}{\omega}(\tfrac{1}{2}\kappa\dddot{D}_{ij}\varepsilon^{ij})_{fi}$$
$$D_{ij} = \mu(r_i r_j - \tfrac{1}{3}\delta_{ij}r^2). \qquad (5.15)$$

D' has been replaced by the traceless form D in (5.15) because of the fact that $\varepsilon^i_i = 0$ (see (2.19)). D_{ij} in equation (5.15) is the quadrupole moment tensor for the material system in the centre-of-mass frame.

It should be appreciated that the equations of motion (5.12) have not been introduced arbitrarily. The amplitude for the radiationless process is uniquely determined to the approximation considered and this yields (5.12) automatically.

Now we derive the analogue of (5.15) for the modified Born approximation case; here it should be remembered that the structure of the Fourier transforms $\psi_\pm(\mathrm{i}\nu, \frac{1}{2}\boldsymbol{p}, \boldsymbol{q})$ selects $\boldsymbol{q}$ arbitrarily close to $\frac{1}{2}\boldsymbol{p}$. Now we go back to (4.9) and recast it in the space–time form in exactly the same way we have done for the Born amplitude. We first recall that $\tilde{\boldsymbol{q}}$ in equation (4.9) is given by

$$\tilde{\boldsymbol{q}} = \boldsymbol{q} - \boldsymbol{q}'.$$

Then using equations (5.6) to (5.9) we recast the amplitude of equation (4.9) into

$$M_{ij} = -\frac{4m_1m_2}{\omega}\int \mathrm{d}t \exp(-\mathrm{i}\omega t)\exp(-\mathrm{i}p^2t/8\mu + ip'^2t/8\mu)$$
$$\times\int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{q}'\,\psi^*_-(-\mathrm{i}\nu', \tfrac{1}{2}\boldsymbol{p}', \boldsymbol{q}')\psi_+(\mathrm{i}\nu, \tfrac{1}{2}\boldsymbol{p}, \boldsymbol{q})$$
$$\times\int \mathrm{d}\boldsymbol{r}\exp[\mathrm{i}(\boldsymbol{q}-\boldsymbol{q}')\cdot\boldsymbol{r}]\tfrac{1}{2}\kappa\dddot{D}_{ij}$$
$$= -\frac{4m_1m_2}{\omega}\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}t\exp(-\mathrm{i}\omega t)\psi_f^*\tfrac{1}{2}\kappa\dddot{D}_{ij}\psi_i \qquad (5.16)$$

where ψ_f, ψ_i are now the Coulomb distorted solutions (4.5) with appropriate boundary conditions; thus we see that the structure of the one-graviton transition operator is unchanged and is essentially $\frac{1}{2}\kappa\dddot{D}_{ij}$.

Our next problem is to relate the matrix elements given by (5.16) and (5.15) to the classical energy-loss formulae; first we do this by comparing results (5.13) and (5.16) with their electrodynamic analogues. In electrodynamics, under identical approximations as made in this paper, the amplitude to emit a photon can be calculated to be

$$M^{\mathrm{e}} = -\frac{4m_1m_2}{\omega}(\boldsymbol{\varepsilon}\cdot\ddot{\boldsymbol{d}})_{fi}$$

where $\boldsymbol{d}$ is the electric dipole moment of the system and $\boldsymbol{\varepsilon}$ the polarisation vector in the gauge ($\varepsilon^0 = 0$). The one-photon transition operator in electrodynamics is $\boldsymbol{\varepsilon}\cdot\ddot{\boldsymbol{d}}$. Since the number of degrees of freedom and the kinematical details are identical for the electromagnetic and gravitational cases, we conclude that the one-graviton transition operator must be $\frac{1}{2}\kappa\dddot{D}_{ij}\varepsilon^{ij}$. The classical energy-loss formula in electrodynamics is

$$\frac{\mathrm{d}E^{\mathrm{e}}}{\mathrm{d}t}=-\frac{(\boldsymbol{\varepsilon}\cdot\ddot{\boldsymbol{d}})^2}{16\pi^2}. \tag{5.18}$$

Hence we conclude that the energy-loss formula for the case of gravitational radiation should be

$$\frac{\mathrm{d}E_{\mathrm{g}}}{\mathrm{d}t}=-\frac{1}{16\pi^2}|\tfrac{1}{2}\kappa\varepsilon_{ij}\dddot{D}_{ij}|^2=-\frac{G}{8\pi}(\varepsilon_{ij}\dddot{D}^{ij})^2. \tag{5.19}$$

This is precisely the formula obtained by Einstein in 1918!

For the reader who is not happy with the arguments given above leading to (5.19), we give here a more elaborate reasoning using arguments similar to the ones found in, say, Landau and Lifshitz. We arrive at the probability for quadrupole gravitational radiation to be (the $4m_1m_2$ factor in the amplitude is removed by the appropriate normalisation factors in the phase-space integrals)

$$\mathrm{d}\omega=\mathrm{d}\Omega_\kappa\frac{|\frac{1}{2}\kappa\varepsilon_{ij}\dddot{D}^{ij}|^2}{4\pi^2\omega}.$$

The intensity of radiation is likewise

$$\frac{\mathrm{d}I^{i\to f}}{d\Omega_\kappa}=\frac{|\frac{1}{2}\kappa\varepsilon_{ij}\dddot{D}_{ij}|^2_{fi}}{4\pi^2}.$$

Now the classical limit is obtained by resorting to the correspondence principle (see Landau and Lifshitz, vol 4, part 1, p 136) which states that the matrix elements $(\frac{1}{2}\kappa\varepsilon_{ij}\dddot{D}_{ij})_{fi}$ are now to be replaced by the value of $\frac{1}{2}\kappa\varepsilon_{ij}\dddot{D}_{ij}$ evaluated for the relevant classical trajectory. Thus

$$\frac{\mathrm{d}E_{\mathrm{g}}}{\mathrm{d}t\,\mathrm{d}\Omega_{\hat{\kappa}}}=-\frac{\frac{1}{2}\kappa(\varepsilon_{ij}\dddot{D}_{ij})^2}{4\pi^2},$$

which is the same result as we had before.

6. Discussions and conclusions

We conclude this paper with a short discussion of its salient features. Inspired by the various criticisms of the existing classical calculations of the energy-loss formula due to gravitational radiation, we have undertaken the study of this problem from the unusual point of view of quantum gravity. Though it may seem unnatural to use quantum methods to study this purely classical situation, we emphasise again that this was done solely due to the cloud of controversies surrounding the classical calculations. We also wish to emphasise the fact that, in principle, there is nothing wrong in approaching even purely classical problems from a quantum point of view as presumably the underlying structure of all classical laws is quantum mechanical. Also, it often happens that quantum derivations are simpler than their classical counterparts, though this is not always true.

We have calculated the one-graviton transition operator by studying the amplitude to emit a graviton during the scattering of two masses m_1 and m_2. We have studied the problem when the masses are scattered both gravitationally and electromagnetically. We find that the two amplitudes are proportional to each other in the non-relativistic limit. By studying the scattering problem, the criterion of 'no incoming radiation' demanded by Ehlers *et al* is naturally satisfied. The emission amplitudes have been calculated in the usual Born approximation as well as in a more exact treatment where the in and out states have been taken to be the confluent hypergeometric functions in place of the plane waves used for the Born approximation. This latter treatment alleviates the criticism levelled against Feynman graph calculations, namely that there exists a potentially large parameter Gm_1m_2/hc in the theory and hence no perturbation theory in the coupling constant should be possible. We find once again that the structure of the one-graviton transition operator is unchanged. By using the confluent hypergeometric wavefunction for the graviton also, the problem of the true light cone versus the flat space light cones has been adequately tackled and we find that because of the non-relativistic nature of the system, no error is caused in using the flat space propagators for the gravitons. The self-interaction of the gravitational field was found to be crucial in obtaining the quadrupole formula: thus the field contribution to the quadrupole formula despite its non-localised nature has been adequately taken into account in our treatment. Our results imply that even though the field contribution is non-localised, the effective size of the system is still governed by the relative separation of the material bodies, which is small compared with the wavelength of the radiation. In the quantum treatment this fact emerges as a strict consequence of the non-relativistic motion of the sources. Thus we feel that we have alleviated the majority of the criticisms cited by Ehlers *et al* and still find the Einstein result for the energy-loss formula due to quadrupole gravitational radiation to be correct. This is in direct contradiction to Rosenblum's claim. We thus maintain that the quadrupole formula should be applicable to the binary pulsar PSR 1913+16. The experimental results from this system also support this view.

Some of the obvious advantages of the quantum approach are (i) there is no need for any renormalisation even if point particles are used. This should be contrasted with the classical calculations where careful renormalisations have to be carried out to remove self-energy ambiguities. (ii) No pseudo-tensor is required as the only relation required to deduce the energy-loss formula is $E = \hbar\omega$.

As a last remark we mention that graviton emission amplitudes have been calculated in the Born approximation by a number of authors before. But these authors have calculated the total cross section and average energy loss in terms of the kinematic invariants of the situation. But it is very hard to isolate the classical energy-loss formula from these results, though admittedly they implicitly contain this in some form. What we have done here instead is to study the non-relativistic form of the emission amplitude and then deduce the quadrupole formula using the correspondence principle and the complementarity between space–time and energy–momentum descriptions. Also our use of the confluent hypergeometric wavefunctions is a significant improvement over the Born approximation.

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